Kinematic Differential Equations

Week-8

ATS 6 Satellite Attitude Axes

- Pitch (To Change Altitude - Satellite Tilts in Azimuth as Seen From Earth)
- Yaw (To Change Orbit Inclination or Latitude - Satellite Rotates About its Centre as Seen From Earth)
- Roll Axis (Satellite Tilts in Elevation as Seen From Earth)
- Direction Along the Orbit

To Polaris the Pole Star

North

West

East

To the Centre of the Earth

Electropaedia
Attitude Kinematics

• In the previous sections, we developed several different ways to describe the attitude, or orientation, of one reference frame with respect to another, in terms of attitude variables.

• In this section, we treat kinematics in which the relative orientation between two reference frames is time dependent.

• The time-dependent relationship between two reference frames is described by so called kinematic differential equations.

• In this section we derive the kinematic differential equations for the:
  – Direction cosine matrix,
  – Euler angles, and
  – Quaternions.
Direction Cosine Matrix

- **A** and **B** are two reference frames moving relative to each other.
- The **angular velocity vector** of reference frame **B** with respect to reference frame **A** is denoted by:
  \[ \omega = \omega_{B/A} \]
  \[ \omega = \omega_1 \mathbf{\hat{b}}_1 + \omega_2 \mathbf{\hat{b}}_2 + \omega_3 \mathbf{\hat{b}}_3 \]
- Where the angular velocity vector \( \omega \) is time dependent.
- We have defined the direction cosine matrix (DCM)

\[
\begin{bmatrix}
\mathbf{\hat{b}}_1 \\
\mathbf{\hat{b}}_2 \\
\mathbf{\hat{b}}_3 \\
\end{bmatrix} = C \\
\begin{bmatrix}
\mathbf{\hat{a}}_1 \\
\mathbf{\hat{a}}_2 \\
\mathbf{\hat{a}}_3 \\
\end{bmatrix} = C^{-1} \\
\begin{bmatrix}
\mathbf{\hat{b}}_1 \\
\mathbf{\hat{b}}_2 \\
\mathbf{\hat{b}}_3 \\
\end{bmatrix} = C^T \\
\begin{bmatrix}
\mathbf{\hat{b}}_1 \\
\mathbf{\hat{b}}_2 \\
\mathbf{\hat{b}}_3 \\
\end{bmatrix}
\]

\[ C \equiv C^{B/A} \]
Direction Cosine Matrix

• Because the two reference frames are rotating relative to each other, the **direction cosine matrix** and its elements $C_{ij}$ are functions of time.

• Taking the time derivative in $A$ and denoting it by an **overdot**, we obtain:

$$
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= \dot{C}^T
\begin{bmatrix}
\vec{b}_1 \\
\vec{b}_2 \\
\vec{b}_3
\end{bmatrix}
+ C^T
\begin{bmatrix}
\dot{\vec{b}}_1 \\
\dot{\vec{b}}_2 \\
\dot{\vec{b}}_3
\end{bmatrix}
= \dot{C}^T
\begin{bmatrix}
\vec{b}_1 \\
\vec{b}_2 \\
\vec{b}_3
\end{bmatrix}
+ C^T
\begin{bmatrix}
\vec{\omega} \times \vec{b}_1 \\
\vec{\omega} \times \vec{b}_2 \\
\vec{\omega} \times \vec{b}_3
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= \dot{C}^T
\begin{bmatrix}
\vec{b}_1 \\
\vec{b}_2 \\
\vec{b}_3
\end{bmatrix}
- C^T
\begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}
\begin{bmatrix}
\vec{b}_1 \\
\vec{b}_2 \\
\vec{b}_3
\end{bmatrix}
$$
Direction Cosine Matrix

- Where

\[ \dot{C} \equiv \begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix} \]

- By defining the skew-symmetric matrix:

\[ \Omega \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \]
Direction Cosine Matrix

• We obtain:

\[
\begin{bmatrix}
\dot{\mathbf{C}}^T - \mathbf{C}^T \mathbf{\Omega}
\end{bmatrix}
\begin{bmatrix}
\mathbf{b}_1 \\
\mathbf{b}_2 \\
\mathbf{b}_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

• We obtain:

\[
\begin{bmatrix}
\dot{\mathbf{C}}^T - \mathbf{C}^T \mathbf{\Omega}
\end{bmatrix}
= 0
\]

• Taking transpose of this equation and \( \mathbf{\Omega}^T = -\mathbf{\Omega} \)

\[
\dot{\mathbf{C}} + \mathbf{\Omega} \mathbf{C} = 0
\]

\[
\dot{\mathbf{C}} = -\mathbf{\Omega} \mathbf{C}
\]
Direction Cosine Matrix

• Which is called the **kinematic differential equations** for the **direction cosine matrix C**.

• Differential equaitons for each element of C can be written as:

\[
\begin{align*}
\dot{C}_{11} &= \omega_3 C_{21} - \omega_2 C_{31} \\
\dot{C}_{12} &= \omega_3 C_{22} - \omega_2 C_{32} \\
\dot{C}_{13} &= \omega_3 C_{23} - \omega_2 C_{33} \\
\dot{C}_{21} &= \omega_1 C_{31} - \omega_3 C_{11} \\
\dot{C}_{22} &= \omega_1 C_{32} - \omega_3 C_{12} \\
\dot{C}_{23} &= \omega_1 C_{33} - \omega_3 C_{13} \\
\dot{C}_{31} &= \omega_2 C_{11} - \omega_1 C_{21} \\
\dot{C}_{32} &= \omega_2 C_{12} - \omega_1 C_{22} \\
\dot{C}_{33} &= \omega_2 C_{13} - \omega_1 C_{23}
\end{align*}
\]

**Derivation: dcm.m**
Direction Cosine Matrix

• If $\omega_1$, $\omega_2$, $\omega_3$ are known as functions of time, then the orientation of $B$ relative to $A$ as a function of time can be determined by solving these kinematic differential equations.

• These equations can be solved using numerical techniques.

• Orthonormality condition is often used to check the accuracy of numerical integration.

• $CC^T=I$

• Example 1: ex1.m
Euler Angles

• Like the kinematic differential equation for the direction cosine matrix \( C \), the orientation of a reference frame \( B \) relative to a reference frame \( A \) can also be described by introducing the time dependence of Euler angles.

• Consider the rotational sequence of \( 3(\theta_3)-2(\theta_2)-1(\theta_1) \) to \( B \) from \( A \). Derivation 1:

\[
\begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3
\end{bmatrix} = \begin{bmatrix}
\dot{\theta}_1 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0
\end{bmatrix} + \begin{bmatrix}
C_1(\theta_1) \\
C_1(\theta_1)C_2(\theta_2)
\end{bmatrix} \begin{bmatrix}
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & -\sin \theta_2 \\
0 & \cos \theta_1 & \sin \theta_1 \cos \theta_2 \\
0 & -\sin \theta_1 & \cos \theta_1 \cos \theta_2
\end{bmatrix} \begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix}
\]

• The inverse relationship

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix} = \frac{1}{\cos \theta_2} \begin{bmatrix}
c \theta_2 & s \theta_1 s \theta_2 & c \theta_1 s \theta_2 \\
0 & c \theta_1 c \theta_2 & -s \theta_1 c \theta_2 \\
0 & s \theta_1 & c \theta_1
\end{bmatrix} \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix}
\]

\[
\omega = S(\theta) \dot{\theta}
\]

\[
\dot{\theta} = S^{-1}(\theta) \omega
\]
Euler Angles

• If $\omega_1$, $\omega_2$, $\omega_3$ are known as functions of time, and have initial conditions for the three Euler angles then the orientation of $B$ relative to $A$ as a function of time can be determined by solving these kinematic differential equations.

• These equations can be solved using numerical techniques.

• However these calculations involve the computation of the trigonometric functions of the angle.

• Singularites exist when $\theta_2=\pi/2$.

• Such a mathematical singularity problem can be avoided by selecting a different set of Euler angles.
Euler Angles

• Similarly for $3(\Phi)-1(\theta)-3(\psi)$ sequence:

$$
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\psi \\
\end{bmatrix} + C_3(\psi)
\begin{bmatrix}
\dot{\theta} \\
0 \\
0 \\
\end{bmatrix} + C_3(\psi)C_1(\theta)
\begin{bmatrix}
0 \\
0 \\
\dot{\phi} \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi} \\
\end{bmatrix} = \frac{1}{\sin \theta}
\begin{bmatrix}
s\psi & c\psi & 0 \\
c\psi s\theta & -s\psi s\theta & 0 \\
-s\psi c\theta & -c\psi c\theta & s\theta \\
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\end{bmatrix}
$$

Example 2: example2.m
Example 3: example3.m
Rotation Vector Kinematics

• Desired kinematic equation for the rotation vector:
  \[
  \mathbf{v} = 2\left(\cos^{-1} q_4\right) \frac{\mathbf{q}}{||\mathbf{q}||}
  \]

  \[
  \dot{\mathbf{\theta}} = \mathbf{\omega} + \frac{1}{2} \mathbf{\theta} \times \mathbf{\omega} + \frac{1}{\mathbf{\theta}^2}\left(1 - \frac{\mathbf{\theta}}{2} \cot \frac{\mathbf{\theta}}{2}\right) \mathbf{\theta} \times (\mathbf{\theta} \times \mathbf{\omega})
  \]

• Inverse is:
  \[
  \mathbf{\omega} = \dot{\mathbf{\theta}} - \frac{1 - \cos \mathbf{\theta}}{\mathbf{\theta}^2} \mathbf{\theta} \times \dot{\mathbf{\theta}} + \frac{\mathbf{\theta} - \sin \mathbf{\theta}}{\mathbf{\theta}^3} \mathbf{\theta} \times (\mathbf{\theta} \times \dot{\mathbf{\theta}})
  \]
Quaternions

In strapdown inertial reference system of aerospace vehicles, the body rates $\omega_1, \omega_2, \omega_3$ are measured by rate gyros that are «strapped down» to the vehicles.

The kinematic differential equations is then integrated numerically using onboard flight computer to determine the orientation of the vehicles in terms of quaternions.

Quaternions have no singularity as do Euler angles.

Moreover, quaternions are well suited for onboard real-time computation because only products and no trigonometric relations exist in the kinematic differential equations.

Thus, S/C orientation is now commonly described in terms of quaternions.

Example 4:exe euler_quat.m

In short form:

$$\dot{q} = \frac{1}{2} (q_4 \omega - \omega \times q)$$

$$\dot{q}_4 = -\frac{1}{2} \omega^T q$$
Rodrigues Parameter Kinematics

- The kinematic equation satisfied by the Gibbs vector is most easily obtained from the kinematic equation for the quaternion.

\[ \dot{\mathbf{g}} = (1/2) \left[ \mathbf{\omega} + \mathbf{g} \times \mathbf{\omega} + (\mathbf{g} \cdot \mathbf{\omega})\mathbf{g} \right] = (1/2) \left( \mathbf{I}_3 + [\mathbf{g} \times] + \mathbf{gg}^T \right) \mathbf{\omega} \]

- The inverse of this equation is:

\[ \mathbf{\omega} = 2 \left( \mathbf{I}_3 + [\mathbf{g} \times] + \mathbf{gg}^T \right)^{-1} \dot{\mathbf{g}} = 2 \left( 1 + \|\mathbf{g}\|^2 \right)^{-1} (\dot{\mathbf{g}} - \mathbf{g} \times \dot{\mathbf{g}}) \]
Modified Rodrigues Parameter Kinematics

• The kinematic equation for the modified Rodrigues parameters can also be obtained from the kinematic equation for the quaternion.

\[
\dot{p} = \frac{1}{4} \left[ (1 - \|p\|^2) \omega + 2p \times \omega + 2(p \cdot \omega)p \right]
= \frac{1}{4} \left\{ (1 - \|p\|^2) I_3 + 2[p\times] + 2pp^T \right\} \omega
= \frac{1 + \|p\|^2}{4} \left( I_3 + 2 \frac{[p\times]^2 + [p\times]}{1 + \|p\|^2} \right) \omega
\]

• The inverse of the kinematic equation for the MRPs is:

\[
\omega = \frac{4}{1 + \|p\|^2} \left( I_3 + 2 \frac{[p\times]^2 - [p\times]}{1 + \|p\|^2} \right) \dot{p}
= 4 (1 + \|p\|^2)^{-2} \left[ (1 - \|p\|^2)\dot{p} - 2p \times \dot{p} + 2(p \cdot \dot{p})p \right]
\]
CASE STUDY

• B frame 1(a(t))-2(b(t))-1(c(t)) rotates with
  – w1=sin(t)*exp(-4*t);
  – w2=sin(2*t)*exp(-4*t);
  – w3=sin(3*t)*exp(-4*t);

• Relative to A inertial frame.
• If initial values are a(0)=1, b(0)=2, c(0)=3
• Calculate a(5), b(5), c(5)